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Extremal Rocket Motion with Maximum Thrust in a Linear Central Field

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The Mayer variational problem of determining optimal trajectories of a rocket moving with maximum thrust in a thin spherical layer within a central Newtonian field is considered. The present work is devoted to the analytical investigation of maximum thrust arcs for a rocket engine with constant exhaust velocity and limited mass flow rate using the necessary conditions of optimality. Many problems of practical interest admit an approximation of the central Newtonian field by a linear central field. It is shown that the linear central field assumption is valid when the thrust vector is nearly tangential to the orbit and the rocket remains in a sufficiently thin spherical layer during the maneuver. When the linear central field approximation is utilized, analytic closed-form expressions are obtained describing optimal maximum thrust space trajectories. The problem of optimal orbit transfer between coplanar circular orbits is considered to illustrate the solution process, which comprises two maximum thrust and three null thrust arcs. It is shown that solutions of the problem are comparable with solutions based on assumptions of instantaneous changes in velocity, including the Hohmann transfer.

Nomenclature

$\boldsymbol{a}$	=	thrust acceleration vector
$B, D, P_1, R_1, P_2, R_2, \alpha, \alpha_1, \alpha_2, a, a_1, a_2$	=	constants of the primer vector
C	=	constant of Hamiltonian
C_{21}, C_{22}	=	integration constants
c	=	exhaust velocity, km/s
e	=	eccentricity
F_1	=	difference of values of integral sine, rad
F_2	=	difference of values of integral cosine, rad
g	=	sea-level gravitational acceleration, km/s^2
H	=	Hamiltonian
$\mathcal{J}$	=	performance functional
k, k_1	=	Schuler frequency, rad/s
m	=	mass of spacecraft, kg
p	=	semilatus rectum, km
$\boldsymbol{r}$	=	radius vector
t_1, τ_1	=	initial time of first and second maximum thrust arcs, s
t_2, τ_2	=	final time of first and second maximum thrust arcs, s
t_3	=	initial time for second maximum thrust arc computed from beginning of maneuver, s
$\boldsymbol{v}$	=	velocity vector
x_i	=	value of parameter x at the ith switching point
β	=	mass flow rate, kg/s
$\Delta g_2, \Delta g_3$	=	gravitational errors at the second and third switching points, km/s^2
Δr	=	difference between magnitudes of radius vector, km
ΔV	=	velocity change, km/s
δr	=	width of spherical layer, km
θ	=	polar angle, rad
$\boldsymbol{\lambda}\,(\lambda_1, \lambda_2, \lambda_3)$	=	primer vector conjugated with velocity
$\boldsymbol{\lambda}_r\,(\lambda_4, \lambda_5)$	=	vector conjugated with the radius vector
λ_7	=	multiplier conjugated with the mass
μ	=	gravitational parameter of central body, km^3/s^2
φ	=	thrust angle (angle between primer vector and horizon), rad
χ	=	switching function
ψ	=	angle between inertial axes and primer vector, rad

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Introduction

OUR objective is to obtain analytic solutions to the problem of computing optimal trajectories, that is, minimal fuel expenditure, of a rocket moving in a Newtonian field and possessing a propulsion system with limited mass flow rate and constant exhaust velocity. It is known that spacecraft trajectories may consist of combinations of null thrust (NT), intermediate thrust (IT), and maximum thrust (MT) arcs.[1] In our work, only combinations of MT and NT arcs are considered. Previous investigations searching for analytic methods applicable to MT arcs include those of Marec,[2] Azimov,[3] and Ehricke.[4] It was shown by Azizov and Korshunova[5] that solving the problem in quadratures requires four first integrals or two integrals in involution for planar motion. For motion in a central Newtonian field, these integrals remain unknown. However, MT arcs can still be studied using 1) numerical techniques or 2) analytical methods based on using either known impulsive solutions or utilizing appropriate approximations. This paper takes the latter approach, utilizing a linear central field approximation. More information on numerical solutions can be found in the works of Jezewski[6] and Hazelrigg et al.[7]

Certain classes of space dynamics problems admit sensible simplifications based on the underlying physics. In such cases, one can sometimes obtain analytical solutions. For example, consider the problem of determining optimal orbit transfers and suppose that the initial and final orbits are circular, but at significantly different altitudes. When considering impulsive maneuvers, the solution is the well-known Hohmann transfer or bielliptical transfer. However, the more realistic situation is that of nonimpulsive maximum thrust maneuvers employing practical propulsion systems. In the case of a central Newtonian field, analytic optimal trajectories are currently not available. Suppose, however, that the optimal transfer trajectory contains two MT arcs connected by an NT arc, for example, an elliptic transfer orbit. If, as with many chemical propulsion systems in use today, the time the engines are thrusting, that is, during the MT arcs, is relatively short, then it might be possible to assume that during the initial and final maneuvers the spacecraft operates in individual thin spherical layers and that the width of these thin layers is small compared to the spacecraft altitude during the maneuvers. Under this assumption it follows that an approximation of the central Newtonian field by a linear central field may be a reasonable approximation during the individual MT arcs. It is shown in this paper that a linear central field approximation leads to closed-form analytic solutions completely describing the MT arcs. When the MT and NT arcs are connected properly, a complete extremal trajectory can be constructed and described via analytic expressions for the spacecraft states and costates as a function of time.

Investigations of linear central fields have been reported by Azizov and Korshunova[5] and Jezewski.[6] In the cited works, it is shown that one difficulty in solving the problem using a linear field approximation is determining a valid initial control vector. To address this difficulty, some authors recommended using basic impulsive solutions as starting points and employing an iterative process to satisfy the terminal conditions and corner conditions.[6,7] Another approach to this problem is to utilize primer vector theory.[1] This paper is devoted to applying primer vector methods to the problem of optimal transfer of a rocket moving with MT in a thin spherical layer within a central Newtonian field.

Linear Field Approximation

The equation of motion for a spacecraft moving in a central Newtonian field is

$$\ddot{\boldsymbol{r}} = -(\mu/r^3)\boldsymbol{r} + \boldsymbol{a} \tag{1}$$

where r is the magnitude of the radius vector $\boldsymbol{r}$ of the spacecraft (represented in a convenient inertial reference frame).

The approximation of the equations of motion involve series expansions of the gravitational term, such that

$$g = \frac{\mu}{r^3} = \frac{\mu}{r_0^3} - 3\frac{\mu}{r_0^4}\Delta r + \frac{1}{2!}\frac{\mu}{r_0^5}(\Delta r)^2 + \cdots + \frac{g^{(n)}(r_0)}{n!}(\Delta_r)^{(n)} + \mathcal{O}[(\Delta r)^n] = \frac{\mu}{r_0^3} + \Delta g \tag{2}$$

where

$$\Delta r := r - r_0, \qquad \Delta g := \frac{\mu}{2r_0^3}\sum_{k=1}^{\infty}(-1)^k(k+1)(k+2)\left(\frac{\Delta r}{r_0}\right)^k$$

Substituting Eq. (2) into Eq. (1) and rearranging terms yields

$$\ddot{\boldsymbol{r}} + \left(\mu/r_0^3\right)\boldsymbol{r} = \boldsymbol{a} - \Delta\boldsymbol{R} \tag{3}$$

where $\Delta\boldsymbol{R} := \Delta g\boldsymbol{r}$.

For problems in which the spacecraft trajectory remains within a thin spherical layer, that is, when Δr remains small, the quantity Δg may be neglected. It follows that, when Δg is neglected, Eq. (1) reduces to the case of the linear central field, wherein

$$\ddot{\boldsymbol{r}} + \left(\mu/r_0^3\right)\boldsymbol{r} = \boldsymbol{a} \tag{4}$$

We can now evaluate the relative error in the dynamic model associated with neglecting $\Delta\boldsymbol{R}$ in Eq. (3) by first writing

$$\epsilon = \left|\frac{1}{r^3} - \frac{1}{r_0^3}\right| = \frac{\Delta r}{r_0^4}\left|\frac{3 + 3(\Delta r/r_0) + (\Delta r/r_0)^2}{1 + 3(\Delta r/r_0) + 3(\Delta r/r_0)^2 + (\Delta r/r_0)^3}\right|$$

Thus, when $\Delta r/r_0 \ll 1$, the relative error associated with Eq. (4) is $\mathcal{O}(\Delta r/r_0^4)$.

According to Lawden,[1] the primer vector is governed by

$$\ddot{\boldsymbol{\lambda}} = \lambda\frac{\partial\boldsymbol{g}}{\partial p} \tag{5}$$

where $\partial\boldsymbol{g}/\partial p$ denotes differentiation with respect to a displacement in the direction $\boldsymbol{\lambda}$. For the case of a central Newtonian field, we have $\boldsymbol{g} = -(\mu/r^3)\boldsymbol{r}$, and we can use Eq. (5) to obtain the primer vector governing equations as

$$\ddot{\boldsymbol{\lambda}} = (3\mu/r^5)(\boldsymbol{\lambda}\cdot\boldsymbol{r})\boldsymbol{r} - (\mu/r^3)\boldsymbol{\lambda} \tag{6}$$

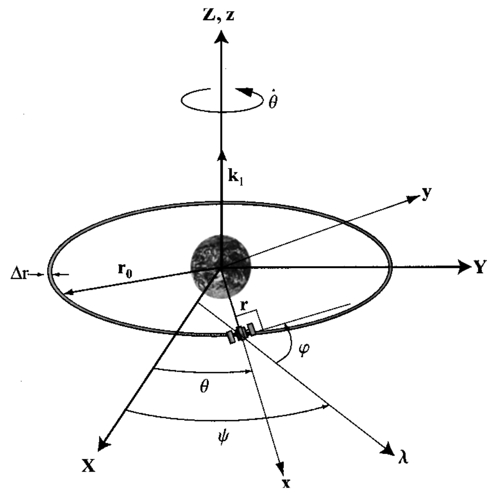

Fig. 1 Primer vector definition and coordinates.

Similarly, in a linear central field, we have $\boldsymbol{g} = -(\mu/r_0^3)\boldsymbol{r}$, and we find that the primer vector is governed by

$$\ddot{\boldsymbol{\lambda}} = -\left(\mu/r_0^3\right)\boldsymbol{\lambda} \tag{7}$$

Using the series expansion for g given in Eq. (2), we rewrite Eq. (6) as

$$\ddot{\boldsymbol{\lambda}} = -\left(\mu/r_0^3\right)\boldsymbol{\lambda} + (3\mu/r^4)\lambda\sin\varphi\boldsymbol{r} - \Delta g\boldsymbol{\lambda} \tag{8}$$

where φ is the angle between $\boldsymbol{\lambda}$ and a line perpendicular to $\boldsymbol{r}$. The coordinate system, defining the angle φ, is depicted in Fig. 1. Comparing Eq. (8) with Eq. (7), and taking into account Δg, one can conclude that a linear approximation of central Newtonian field is admissible if

$$\sin\varphi \approx 0, \qquad \Delta r/r_0 \ll 1 \tag{9}$$

When $\sin\varphi \approx 0$, the second term on the right-hand side of Eq. (8) is negligible. In this situation, the thrust vector is nearly tangential to the orbit. When $\Delta r/r_0 \ll 1$ and Δg is negligible, the third term on the right-hand side of Eq. (8) is negligible.

When solving problems connected with the minimization of fuel expenditure, it is important to evaluate the errors in the minimization functional due to the approximations made in assuming a linear central field. For such problems, the performance functional can be represented in the form

$$\mathcal{J} = c\ln\frac{m_0}{m_1} = \int_0^T |\boldsymbol{a}(t)|\,\mathrm{d}t \tag{10}$$

where m_0 and m_1 are the initial and final rocket masses, respectively, and c is the exhaust velocity (assumed constant). Utilizing Eqs. (3) and (4) in conjunction with Eq. (10) to compute the difference in the functionals corresponding to using the linear central field and the central Newtonian field yields the inequality

$$\Delta\mathcal{J} \leq \int_0^T |\Delta\boldsymbol{R}|\,\mathrm{d}t$$

Define the quantities a_n and S as

$$a_n := (-1)^n(n+1)(n+2)\left(\frac{\Delta r}{r_0}\right)^n$$

$$S := \sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}(-1)^n(n+1)(n+2)\left(\frac{\Delta r}{r_0}\right)^{n-1}$$

It then follows that

$$|\Delta g| = \left(\mu/2r_0^3\right)(\Delta r/r_0)S$$

It can be shown that

$$\lim_{n\to\infty}|a_{n+1}/a_n| = (\Delta r/r_0)\lim_{n\to\infty}[1+2/(n+2)] = \Delta r/r_0$$

and if $\Delta r \ll r_0$, we have

$$\lim_{n\to\infty}|a_{n+1}/a_n| < 1$$

Then, according to D'Alambert's convergence test, the series

$$\sum_{n=1}^{\infty} a_n$$

converges,[8] and

$$|a_1| > |a_2| > |a_3| > \cdots |a_n| > \cdots$$

with $a_n \to 0$ as $n\to\infty$. Consequently, the sum S is bounded above by a finite value, which is denoted here by L. Because $\Delta\boldsymbol{R} = \Delta g\boldsymbol{r}$, we have

$$|\Delta\boldsymbol{R}| \le \tilde{L}(\Delta r/r_0)(1+\Delta r/r_0) \tag{11}$$

where

$$\tilde{L} := \left(\mu/2r_0^2\right)L$$

Therefore, the error in the performance function due to the linear central field approximation is bounded above by

$$\Delta\mathcal{J} \le \int_0^T |\Delta\boldsymbol{R}|\,\mathrm{d}t \le \tilde{L}\frac{\Delta r}{r_0}\left(1+\frac{\Delta r}{r_0}\right)\int_0^T \mathrm{d}t$$

$$= \tilde{L}\frac{\Delta r}{r_0}\left(1+\frac{\Delta r}{r_0}\right)T = \mathcal{O}\left(\frac{\Delta r}{r_0}\right) \tag{12}$$

We conclude that the relative error resulting from ignoring $\Delta\boldsymbol{R}$ is $\mathcal{O}(\Delta r/r_0)$.

In summary, a linear approximation of the central Newtonian field is admissable if the rocket trajectory remains within the boundaries of a thin spherical layer, the width Δr of which is small compared to the magnitude of the radius r_0 and $\sin\varphi \approx 0$. The linear central field approximation leads to a relative error on the order $\mathcal{O}(\Delta r/r_0^4)$ in the dynamic model and relative error on the order $\mathcal{O}(\Delta r/r_0)$ in computation of the performance function.

Canonical Equations and First Integrals for MT Arcs

The canonical system of equations for a rocket in a linear central field are given by[3,5]

$$\dot{\boldsymbol{r}} = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}} = (c\beta/m)(\boldsymbol{\lambda}/\lambda) - k^2\boldsymbol{r}, \qquad \dot{m} = -\beta$$

$$\dot{\boldsymbol{\lambda}} = -\boldsymbol{\lambda}_r, \qquad \dot{\boldsymbol{\lambda}}_r = k^2\boldsymbol{\lambda}, \qquad \dot{\lambda}_7 = (c\beta/m^2)\lambda \tag{13}$$

The Hamiltonian is

$$H = -k^2(\boldsymbol{\lambda}\cdot\boldsymbol{r}) + (\boldsymbol{\lambda}_r\cdot\boldsymbol{v}) + \beta\chi \tag{14}$$

where the switching function[1] is defined to be

$$\chi := (c/m)\lambda - \lambda_7$$

In a polar reference frame (for the planar case), we use the coordinates (r,θ), as depicted in Fig. 1. Let $\lambda_i, i = 1, 2$, denote the components of the primer vector $\boldsymbol{\lambda}$, where $\lambda := \|\boldsymbol{\lambda}\| \ne 0$ for MT arcs, except for cases when a reversal of thrust direction occurs. Define the vector $\boldsymbol{\lambda}_r$, conjugated to the radius vector, as

$$\boldsymbol{\lambda}_r = [\lambda_4 \quad \lambda_1(v_2/r) - \lambda_2(v_1/r) + \lambda_5/r]^T \tag{15}$$

Denote the multipliers conjugated to r and θ as λ_4 and λ_5, respectively, and λ_7 as the multiplier conjugated to the mass m. In planar polar coordinates, the canonical system in Eq. (13) can be expanded, yielding

$$\dot{r} = v_1, \qquad \dot{\theta} = v_2/r, \qquad \dot{v}_1 = (c\beta/m)(\lambda_1/\lambda) - k^2 r + \left(v_2^2/r\right)$$

$$\dot{v}_2 = (c\beta/m)(\lambda_2/\lambda) - (v_1v_2/r), \qquad \dot{m} = -\beta$$

$$\dot{\lambda}_1 = \lambda_2(v_2/r) - \lambda_4, \qquad \dot{\lambda}_2 = -2\lambda_1(v_2/r) + \lambda_2(v_1/r) - (\lambda_5/r)$$

$$\dot{\lambda}_4 = \lambda_1\left[\left(v_2^2/r^2\right) + k^2\right] - \lambda_2\left(v_1v_2/r^2\right) + \lambda_5\left(v_2/r^2\right)$$

$$\dot{\lambda}_5 = 0, \qquad \dot{\lambda}_7 = (c\beta/m^2)\lambda \tag{16}$$

and the Hamiltonian in Eq. (14) reduces to

$$H = \lambda_1\left[\left(v_2^2/r\right) - k^2 r\right] - \lambda_2(v_1v_2/r) + \lambda_4 v_1 + \lambda_5(v_2/r) + \beta\chi$$

From the canonical system in Eq. (13), we determine that the primer vector is governed by

$$\ddot{\boldsymbol{\lambda}} = -k^2\boldsymbol{\lambda}$$

and, in (λ,ψ) coordinates, has projections

$$\ddot{\lambda} - \lambda\dot{\psi}^2 = -k^2\lambda, \qquad 2\dot{\lambda}\dot{\psi} + \lambda\ddot{\psi} = 0 \tag{17}$$

There exist four first integrals associated with the primer vector projections in Eq. (17). Two of the first integrals are

$$\lambda^2\dot{\lambda}^2 = \lambda^2\nu^2 - k^2\lambda^4 - \sigma^2, \qquad \lambda^2\dot{\psi} = \sigma \tag{18}$$

When $\dot{\psi} \ne 0$, we have the additional first integrals

$$\lambda^2 = q/[1 + e\cos 2(\psi - \zeta)]$$

$$kt = \arctan\left[\sqrt{(1-e)/(1+e)}\tan(\psi - \zeta)\right] - \gamma \tag{19}$$

The variables ν, σ, γ, and ζ are independent integration constants, and q and e are related to ν and σ via

$$(1-e^2)\sigma^2 = k^2q^2, \qquad (1-e^2)\nu^2 = 2k^2q \tag{20}$$

From Eq. (18), it follows that, when $\dot{\psi} = 0$, we have $\sigma = 0$, and γ, ζ, q, and e play no role in the solution.

The first integrals in Eqs. (18) and (19) define hodographs of the primer vector for the MT arcs. When the quantity e is depended on, the hodograph of the primer vector is a central ellipse ($e < 1$), a straight line ($e = 1$), or a circle ($e = 0$). It can be shown that choosing the central ellipse leads to analytical closed-form solutions representing motion along circular trajectories. In this paper, the primer vector trajectories are taken to be straight lines. Three other first integrals associated with the canonical system are found to be

$$\lambda_1\left[\left(v_2^2/r\right) - k^2 r\right] - \lambda_2(v_1v_2/r) + \lambda_4 v_1 + \lambda_5(v_2/r) + \chi\beta = C$$

$$m = m_0 - \beta t, \qquad \lambda_5 = \lambda_{50} \tag{21}$$

where C, m_0, and λ_{50} are independent integration constants.

Consider a fixed Cartesian coordinate system *OXYZ* and a coordinate system *Oxyz* that is rotating with angular velocity $\dot{\theta}$ with respect to the *OXYZ*, as illustrated in Fig. 1. If $\dot{\boldsymbol{\lambda}}$ is a derivative of $\boldsymbol{\lambda}$ in the *OXYZ* system, and $\hat{\boldsymbol{\lambda}}$ is a derivative of $\boldsymbol{\lambda}$ in the *Oxyz* system, then

$$\dot{\boldsymbol{\lambda}} = \check{\boldsymbol{\lambda}} + \dot{\theta}\boldsymbol{k}_1 \times \boldsymbol{\lambda} \tag{22}$$

In (r,θ) coordinates, the rocket equations of motion are

$$\ddot{r} - \dot{\theta}^2 r = c\beta\lambda_1/m\lambda - k^2 r, \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = c\beta\lambda_2/m\lambda \tag{23}$$

Introducing the transformation $\boldsymbol{\lambda} = r\boldsymbol{q}$, and with the definition

$$\boldsymbol{q}' := \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\theta}$$

we compute the derivatives

$$\check{\boldsymbol{q}} = \dot{\theta}\boldsymbol{q}', \qquad \frac{\mathrm{d}\check{\boldsymbol{q}}}{\mathrm{d}t} = \dot{\theta}^2\boldsymbol{q}'' + \ddot{\theta}\boldsymbol{q}'$$

Using Eqs. (22) and (23) and the derivatives just defined, we obtain

$$\ddot{\boldsymbol{\lambda}} = r\dot{\theta}^2\boldsymbol{q}'' + (c\beta/m)(\lambda_2/\lambda)\boldsymbol{q}' + \left(c\beta\lambda_1/m\lambda - k^2 r\right)\boldsymbol{q}$$

$$+ (c\beta\lambda_2/m\lambda)\boldsymbol{k}_1 \times \boldsymbol{q} + 2r\dot{\theta}^2\boldsymbol{k}_1 \times q' + r\dot{\theta}^2\boldsymbol{k}_1\,(\boldsymbol{q}\cdot\boldsymbol{k}_1) \qquad (24)$$

With $\boldsymbol{q} = \boldsymbol{q}(u_1, u_2)$, it follows that $\lambda_1 = ru_1$ and $\lambda_2 = ru_2$, and, using Eq. (24), the projections of $\ddot{\boldsymbol{\lambda}}$ on the polar coordinate system axes are

$$(\ddot{\boldsymbol{\lambda}})_r = r\dot{\theta}^2 u_1'' + (c\beta/m\lambda)ru_2u_1' + (c\beta/M\lambda)ru_1^2 - k^2 ru_1$$

$$- (c\beta/m\lambda)ru_2^2 - 2r\dot{\theta}^2u_2' = -k^2\lambda_1$$

$$(\ddot{\boldsymbol{\lambda}})_\theta = r\dot{\theta}^2 u_2'' + (c\beta/m\lambda)ru_2u_2' + (c\beta/m\lambda)ru_1u_2 - k^2 ru_2$$

$$+ (c\beta/m\lambda)ru_1u_2 + 2r\dot{\theta}^2u_1' = -k^2\lambda_2 \qquad (25)$$

When the relationships given in Eq. (25), together with the integral for the Hamiltonian given by the first expression in Eq. (21), are utilized, it can be shown that

$$\dot{\theta}^2(u_1'' - 2u_2') + (c\beta/m\lambda)\left(u_2u_1' + u_1^2 - u_2^2\right) = 0$$

$$\dot{\theta}^2(u_2'' + 2u_1') + (c\beta/m\lambda)(u_2u_2' + 2u_1u_2) = 0 \qquad (26)$$

The first integral associated with Eq. (26) is

$$u_1^2 + u_2^2 + 2C_1u_2 = 0 \qquad (27)$$

where $C_1 \neq 0$ is an integration constant. Using that

$$\lambda_1 = \lambda\sin\varphi, \qquad \lambda_2 = \lambda\cos\varphi \qquad (28)$$

substituting $\lambda_1 = ru_1$ and $\lambda_2 = ru_2$ into Eq. (27), and simplifying yields

$$r = \lambda C_2/\cos\varphi \qquad (29)$$

where $C_2 = -1/2C_1$. Note that a singularity occurs in Eq. (29) when $\cos\varphi = 0$. However, if $\cos\varphi = 0$, then the linear field assumption, that is, $\sin\varphi \approx 0$, is violated. Therefore, it is reasonable in our case to assume that $\cos\varphi \neq 0$. Also, $\cos\varphi = 0$ implies that the rocket is thrusting radially from the central attracting body, a scenario not considered in this paper.

Using that $\dot{r} = v_1$, $r\dot{\theta} = v_2$, $\lambda_2 = \lambda\cos\varphi$, and

$$\varphi = (\pi/2) + \theta - \psi \qquad \text{(which implies} \qquad \dot{\varphi} = \dot{\theta} - \dot{\psi})$$

we take the time derivative of Eq. (29) and substitute appropriately into the equation for $\dot{\lambda}_2$ from the canonical system given in Eq. (16), yielding the important relationship between λ_5 and $\dot{\psi}$:

$$\lambda_5 = -2r\lambda\sin\varphi\dot{\psi} \qquad (30)$$

Utilizing Eqs. (18) and (28), we find that Eq. (30) can also be written as

$$\lambda_5 = -(2\sigma r\sin\varphi/\lambda) \qquad (31)$$

Analytic Solutions for Maximum Thrust Arcs

In this section, we present the analytic solutions to the problem of optimal motion in a linear central field. The solutions are general in the sense that the particular performance functional $\mathcal{J}$ is not specified.

Using the expressions in Eqs. (16), (18), (28), (31), and that

$$\dot{\varphi} = \dot{\theta} - \dot{\psi} = v_2/r - \sigma/\lambda^2$$

we obtain the relationship

$$\lambda_2 v_1 - \lambda_1 v_2 - \lambda_5 - \left(\lambda_2/\lambda^2\right)rz - \sigma\left(\lambda_1/\lambda^2\right)r = 0 \qquad (32)$$

where we define

$$z := \sqrt{\lambda^2\nu^2 - k^2\lambda^4 - \sigma^2}$$

Similarly, starting with the expression for $\dot{\lambda}_1$ from the canonical system given in Eq. (16), namely,

$$\dot{\lambda}_1 = \lambda_2(v_2/r) - \lambda_4$$

we make appropriate substitutions to obtain the relationship

$$\lambda_4 = \sigma\left(\lambda_2/\lambda^2\right) - \left(\lambda_1/\lambda^2\right)z \qquad (33)$$

Using the Eqs. (29), (32), and (33), we rewrite the integral for the Hamiltonian [the first expression in Eq. (21)] as

$$\frac{\mathrm{d}}{\mathrm{d}t}(C_2\lambda\tan\varphi) + \left[\frac{\lambda^4k^2 + \sigma^2}{\lambda^2 z}\right](C_2\lambda\tan\varphi) - \left[\frac{\lambda(\chi\beta - C)}{z}\right] = 0$$

which can be integrated in quadratures yielding $\tan\varphi$:

$$\tan\varphi = (h_1 + Q)/C_2h_2\lambda$$

where

$$h_1 = -\int_0^t \frac{\lambda(\chi\beta - C)h_2}{z}\,\mathrm{d}t, \qquad h_2 = e^{h_3}$$

$$h_3 = \int_0^t \frac{\lambda^4k^2 + \sigma^2}{\lambda^2 z}\,\mathrm{d}t, \qquad \chi = \int_0^t \frac{c}{m\lambda}z\,\mathrm{d}t + \chi_0 \qquad (34)$$

and where σ, C, C_2, χ_0, and Q are integration constants. Collecting all of the relevant equations, the general solution is found to be

$$v_1 = r\left[\frac{z}{\lambda^2} + d\tan\varphi\right], \qquad v_2 = r\left[\frac{\sigma}{\lambda^2} + d\right]$$

$$r = \frac{\lambda C_2}{\cos\varphi}, \qquad \theta = \varphi + \psi - \frac{\pi}{2}, \qquad m = m_0 - \beta t$$

$$\lambda_1 = \lambda\sin\varphi, \qquad \lambda_2 = \lambda\cos\varphi, \qquad \lambda_4 = \sigma\frac{\lambda_2}{\lambda^2} - \frac{\lambda_1}{\lambda^2}z$$

$$\lambda_5 = \lambda_{50}, \qquad \lambda_7 = c\beta\int_0^t \frac{\lambda}{m^2}\,\mathrm{d}\tau + \lambda_{70} \qquad (35)$$

where

$$d = \left(\frac{\chi\beta - C}{C_2}\cos^2\varphi - \frac{\nu^2}{2}\sin 2\varphi\right)\frac{1}{z}$$

and where m_0, λ_{50}, ν, and λ_{70} are integration constants. The values of the various constants are problem specific and depend on the boundary conditions and on the selected performance index. Note that the variables λ and ψ are explicit functions of time [see Eqs. (18) and (19)]. The expressions (35) are the solutions of the canonial system of equations (24) and describe MT arcs in the linear central field regardless the functional of the problem or whether time is fixed or free.

In the particular case of free final time (when $C = 0$) and minimum fuel, that is, $\mathcal{J} = m_0 - m_1$, it follows from the transversality condition[1] that

$$\lambda_{51} = -\frac{\partial\mathcal{J}}{\partial\theta_1} = 0$$

where the subscript 1 denotes the value at the final time. According to the first integrals of the canonical system of equations [see Eq. (21)], we know that λ_5 is a constant, and, according to the Weierstrass condition, it has to be continuous over the entire trajectory.[1] Therefore, we have

$$\lambda_5 = \lambda_{50} = \lambda_{51} = 0$$

Recalling Eq. (31), we obtain the important result that

$$\sigma\sin\varphi \equiv 0$$

Thus, we can consider two cases: 1) $\sigma \equiv 0$ and $\sin\varphi \neq 0$ or 2) $\sin\varphi \equiv 0$ and $\sigma \neq 0$. If $\sigma \neq 0$ and $\sin\varphi \equiv 0$, we have the case of tangential thrust, which, although representing a problem of practical interest, is not considered in this paper. Accepting that $\sin\varphi \neq 0$ and $\sigma \equiv 0$, we find from Eq. (18) that $\sigma = 0$ implies $\dot{\psi} = 0$ (because $\lambda \neq 0$), and hence,

$$\psi = \psi_0$$

where ψ_0 is a constant. With $\dot{\psi} = 0$, we can solve Eq. (17) for λ, yielding

$$\lambda = a\sin(kt + \alpha)$$

where a and α are integration constants, and, in terms of ν in Eq. (18), we have the relationship $\nu = ak$. That $\dot{\psi} = 0$ implies that the hodograph of the primer vector is a straight line and the direction of thrust is inertially fixed. We make the following definitions:

$$x := (km_0/\beta) - kt, \qquad x_0 := km_0/\beta, \qquad \alpha_0 := \alpha + (km_0/\beta)$$

Then, with $\dot{\psi} = 0$ and $\lambda = a\sin(kt + \alpha)$, the switching function is

$$\begin{aligned}
\chi &= akc\int_0^t \frac{\cos(kt+\alpha)}{m_0 - \beta t}\,\mathrm{d}t \\
&= \frac{akc}{\beta}\left[\int_{x_0}^{x} \frac{\sin(\alpha_0)\sin(x) - \cos(\alpha_0)\cos(x)}{x}\,\mathrm{d}x\right] \\
&= \frac{akc}{\beta}\{\sin(\alpha_0)[Si(x) - Si(x_0)] - \cos(\alpha_0)[Ci(x) - Ci(x_0)\} \\
&= -\frac{akc}{\beta}[F_1(x_0, x)\sin(\alpha_0) + F_2(x_0, x)\cos(\alpha_0)] \qquad (36)
\end{aligned}$$

where

$$F_1 := F_1(x_0, x) = Si(x) - Si(x_0)$$

$$F_2 := F_2(x_0, x) = Ci(x) - Ci(x_0)$$

and the functions $Si(x)$ and $Ci(x)$ are integral sine and integral cosine functions:

$$Si(x) := \sum_{i=1}^{\infty} \frac{(-1)^{i+1}x^{2i-1}}{(2i-1)(2i-1)!}$$

$$Ci(x) := C_0 + ln(x) + \sum_{i=1}^{\infty} \frac{(-1)^i x^{2i}}{(2i)(2i)!}$$

where $C_0 = 0.577216$ is the Euler–Mascheroni constant (see Ref. 8).

The complete solution to the free final time, minimum fuel problem, obtained from the general solution described in Eq. (35), is given by

$$v_1 = r[k\cot(kt+\alpha) + \dot{\varphi}\tan\varphi]$$

$$v_2 = -aC_2k\frac{\sin\varphi}{\cos(kt+\alpha)} + \frac{\chi\beta}{ak}\frac{2\cos^2\varphi}{aC_2\sin 2(kt+\alpha)}$$

$$r = aC_2\frac{\sin(kt+\alpha)}{\cos\varphi}, \qquad \theta = \varphi + \psi_0 - \frac{\pi}{2}$$

$$m = m_0 - \beta t, \qquad \lambda_1 = a\sin(kt+\alpha)\sin\varphi$$

$$\lambda_2 = a\sin(kt+\alpha)\cos\varphi, \qquad \lambda_4 = -ak\cos(kt+\alpha)\sin\varphi$$

$$\lambda_7 = acm_0^2\left[\frac{\sin(kt+\alpha)}{m_0 - \beta t}\right] - \chi + \lambda_{70} \qquad (37)$$

where

$$\tan\varphi = \frac{\tan\alpha\tan\varphi_0}{\tan(kt+\alpha)} + \frac{\chi\beta}{ak}\frac{1}{aC_2k} + \frac{cs}{aC_2k\tan(kt+\alpha)}$$

$$\dot{\varphi} = -\frac{k\sin 2\varphi}{\sin^2(kt+\alpha)} + \frac{\chi\beta}{ak}\frac{2\cos^2\varphi}{aC_2\sin 2(kt+\alpha)}$$

$$s = F_2\sin\alpha_0 - F_1\cos\alpha_0$$

and φ_0 is a new integration constant. Expressions (37) are the closed-form analytical solutions of the canonical equations (24) for the free final time problem and describe motion along the MT arcs. We note that in our problem statement the mass flow rate is limited. Therefore, our solution is not equivalent to the case of instantaneous velocity change, which, as pointed out by Battin,[9] is generally an inadequate assumption if one is interested in developing rocket guidance solutions. The last expression of Eq. (37) can be modified to represent an approximate optimal rocket engine steering law in a realistic gravitational field. This is an extension of the idea of employing a linear-tangent steering law that can be obtained by using constant gravity vector.[9]

Transfer Between Circular Orbits

In this section, we apply the closed-form analytical solution given in Eq. (37) that describes the minimum fuel trajectory along an MT arc in a linear central gravity field to the problem of optimal free final time transfer between concentric circular orbits in a central Newtonian field. The problem scenario is illustrated in Fig. 2. The trajectory comprises five separate arcs: NT–MT–NT–MT–NT. In this situation, the first and last NT arcs are the initial and final circular orbits, and the center NT arc is assumed to be a Keplerian elliptical transfer orbit. The spacecraft is transferred to the elliptic orbit with parameters e, p, and ω from the initial orbit by the first MT arc. The second MT arc transfers from the elliptical orbit to the final circular orbit. Consequently, the optimal trajectory must contain four switching points, as depicted in Fig. 2. The first switching point is on the initial circular orbit, the second and third switching points are on the elliptic transfer orbit, and the fourth switching point is on the final circular orbit. The assumption of a linear central gravity field applies only in the two individual thin layers around the initial and final orbits, that is, only during the MT arcs that coincide with the time that the propulsion system is operating. The performance function that we minimize is the difference between the initial and final mass, that is,

$$\mathcal{J} = m_0 - m_1$$

The initial conditions are given as

$$t = t_1 = 0, \qquad r = r_0, \qquad v_{11} = 0, \qquad v_{21} = kr_0, \qquad m = m_0$$

and the final conditions are

$$t = t_2, \qquad r = r_1(r_1 \neq r_0), \qquad v_{12} = 0, \qquad v_{22} = k_1r_1$$

where $k = \sqrt{(\mu/r_0^3)}$ and $k_1 = \sqrt{(\mu/r_1^3)}$ and v_{11} and v_{12} are the radial velocity components at the initial and final times, respectively, and v_{21} and v_{22} are the transversal velocity components at the initial and final times, respectively. The phasing on the initial and final orbits is assumed arbitrary.

According to the theory of optimal trajectories, at the switching points, the radius, velocity, and primer vectors must all be continuous, and the switching function χ must vanish.[1] We will utilize $\chi = 0$ to obtain expressions for α_0 and s at the second and fourth switching points. Using the formulas given in Eq. (37) for the first switching point, we obtain

$$0 = \varphi_1 + \psi_1 - \frac{\pi}{2}, \qquad r_0 = a_1C_{21}\frac{\sin\alpha_1}{\cos\varphi_1}$$

$$kr_0 = a_1C_{21}\frac{\sin\alpha_1}{\cos\varphi_1}\dot{\varphi}_1, \qquad P_1\sin f_1 = a_1\sin\alpha_1\sin\varphi_1$$

$$2P_1\cos f_1 + R_1 = a_1\sin\alpha_1\cos\varphi_1$$

$$P_1\cos f_1 + R_1 = a_1\cos\alpha_1\sin\varphi_1 \qquad (38)$$

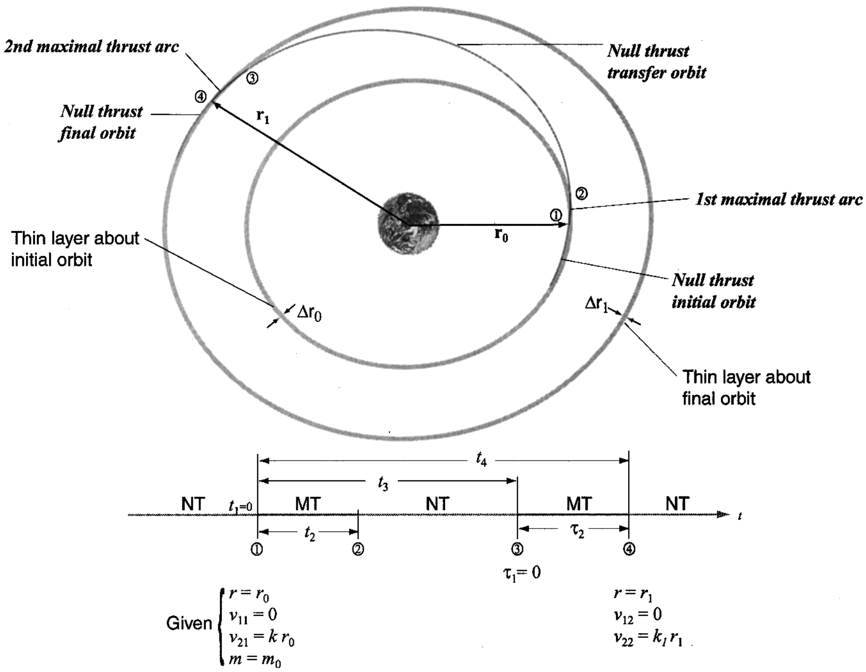

Fig. 2 Transfer between two circular orbits with maximum thrust.

where f_1 is the true anomaly; $\theta_1 = 0$ is the polar angle of the initial point of the first MT arc, which can be set to zero due to the assumption of arbitrary phasing on the initial orbit; and

$$\dot{\varphi}_1 = -\frac{k \sin 2\varphi_1}{\sin 2\alpha_1} \tag{39}$$

On the basis of the properties of the switching function at the switching points, and using expressions for the radius, velocity, and primer vectors for motion on elliptic orbit NT arcs and on MT arcs, we obtain the following conditions at the second switching point:

$$F_1 \sin\alpha_3 + F_2 \cos\alpha_3 = 0, \qquad \theta_2 = \varphi_2 + \psi_2 - \frac{\pi}{2} = f_2 + \omega$$

$$\frac{p}{1 + e\cos f_2} = a_1 C_{21} \frac{\sin(kt_2 + \alpha_1)}{\cos\varphi_2}$$

$$\sqrt{\frac{\mu}{p}} e \sin f_2 = a_1 C_{21} \frac{\sin(kt + \alpha_1)}{\cos\varphi_2} [k \cot(kt + \alpha_1) + \dot{\varphi}_2 \tan\varphi_2]$$

$$\sqrt{\frac{\mu}{p}} (1 + e\cos f_2) = a_1 C_{21} \frac{\sin(kt_2 + \alpha_1)}{\cos\varphi_2} \dot{\varphi}_2$$

$$Be \sin f_2 = a_1 \sin(kt_2 + \alpha_1) \sin\varphi_2$$

$$B(1 + e\cos f_2) + \frac{D}{1 + e\cos f_2} = a_1 \sin(kt_2 + \alpha_1)\cos\varphi_2$$

$$\psi_2 = \psi_1 \tag{40}$$

where

$$\tan\varphi_2 = \frac{\tan\varphi_1 \tan\alpha_1}{\tan(kt_2 + \alpha_1)} + \frac{cs_1}{a_1 C_{21} k \tan(kt_2 + \alpha_1)}$$

$$\dot{\varphi}_2 = -\frac{k \sin 2\varphi_2}{\sin 2(kt_2 + \alpha_1)}, \qquad s_1 = F_2 \sin\alpha_3 - F_1 \cos\alpha_3$$

$$\alpha_3 = \alpha_1 + \frac{km_0}{\beta}, \qquad F_1 = Si(x_2) - Si(x_1)$$

$$F_2 = Ci(x_2) - Ci(x_1)$$

$$x_1 = \frac{km_0}{\beta}, \qquad x_2 = \frac{km_0}{\beta} - kt_2$$

and where (p, e, ω) are parameters of the elliptic transfer orbit, t_1 and t_2 are the initial and final times on the first MT arc, respectively; and $\psi_2 = \psi_1$ because ψ is a constant. Without loss of generality, we can choose $t_1 = 0$, and doing so allows us to say that the total maneuver time, that is, the time the propulsion system is operating, for the first MT arc is given by t_2.

We introduce a new variable τ to denote the total time of motion on the second MT arc that satisfies the inequality $\tau_1 < \tau < \tau_2$, where τ_1 and τ_2 are the initial and final time of the motion on the second MT arc. Without loss of generality, we can choose $\tau_1 = 0$, and doing so allows us to say that the total maneuver time for the second MT arc is given by τ_2. The variable τ is related to the total current flight time t by $\tau = t - t_3$, where $t \geq t_3$ and t_3 is the final time on the elliptic transfer orbit, that is, t_3 is the initial time of the second MT arc, and is computed from the beginning of the first MT arc at $t_1 = 0$. For the final time τ_2, we have $\tau_2 = t_4 - t_3$, where t_4 is the final time of motion on the second MT arc and computed from $t_1 = 0$ (see Fig. 2).

At the third switching point, the following conditions must be satisfied:

$$\theta_3 = \varphi_3 + \psi_3 - \frac{\pi}{2} = f_3 + \omega, \qquad \frac{p}{1 + e\cos f_3} = a_2 C_{22} \frac{\sin\alpha_2}{\cos\varphi_3}$$

$$\sqrt{\frac{\mu}{p}} e \sin f_3 = a_2 C_{22} \frac{\sin\alpha_2}{\cos\varphi_3} [k \cot\alpha_2 + \dot{\varphi}_3 \tan\varphi_3]$$

$$\sqrt{\frac{\mu}{p}} (1 + e\cos f_3) = a_2 C_{22} \frac{\sin\alpha_2}{\cos\varphi_3} \dot{\varphi}_3$$

$$Be \sin f_3 = a_2 \sin\alpha_2 \sin\varphi_3$$

$$B(1 + e\cos f_3) + \frac{D}{1 + e\cos f_3} = a_2 \sin\alpha_2 \cos\varphi_3$$

$$a_1 \sin(k_1 t_2 + \alpha_1) = a_2 \sin\alpha_2 \tag{41}$$

where

$$\dot{\varphi}_3 = -\frac{k1 \sin 2\varphi_3}{\sin^2\alpha_2} \tag{42}$$

At the fourth switching point we have

$$F_3 \sin\alpha_4 + F_4 \cos\alpha_4 = 0, \qquad \theta_4 = \varphi_4 + \psi_4 - \frac{\pi}{2}$$

$$r_1 = a_2 C_{22} \frac{\sin(k_1\tau_2 + \alpha_2)}{\cos\varphi_4}, \qquad \sqrt{\frac{\mu}{r_1}} = a_2 C_{22} \frac{\sin(k_1\tau_2 + \alpha_2)}{\cos\varphi_4} \dot{\varphi}_4$$

$$P_2 \sin f_4 = a_2 \sin(k_1\tau_2 + \alpha_2) \sin\varphi_4$$

$$P_2 \cos f_4 + R_2 = a_2 \cos\alpha_2 \sin\varphi_4$$

$$2P_2 \cos f_4 + R_2 = a_2 \sin(k_1\tau_2 + \alpha_2)\cos\varphi_4, \qquad \psi_4 = \psi_3 \tag{43}$$

where

$$\tan\varphi_4 = \frac{\tan\varphi_3\tan\alpha_2}{\tan(k_1\tau_2+\alpha_2)} + \frac{cs_2}{a_2C_{22}k_1\tan(k_1\tau_2+\alpha_2)}$$

$$\dot{\varphi}_4 = -\frac{k1\sin2\varphi_4}{\sin2(k_1\tau_2+\alpha_2)}, \qquad s_2 = F_4\sin\alpha_4 - F_3\cos\alpha_4$$

$$\alpha_4 = \alpha_2 + \frac{k_1(m_0-\beta t_2)}{\beta}, \qquad F_3 = Si(x_4) - Si(x_3)$$

$$F_4 = Ci(x_4) - Ci(x_3), \qquad x_3 = \frac{k_1 m_0}{\beta} - k_1 t_2$$

$$x_4 = \frac{k_1 m_0}{\beta} - k_1(t_2+\tau_2) \tag{44}$$

The transversality condition (taking into account the arbitrary phasing on the boundary orbits) is

$$\lambda_{71} = -\frac{\partial\mathcal{J}}{\partial m_1} = 1 \tag{45}$$

at τ_2. Consequently, at the fourth switching point we have

$$ca_2\sin(k_1\tau_2+\alpha_2) = c\lambda_0 = m_0 - \beta(t_2+\tau_2) \tag{46}$$

The flight time between the second and third switching points is determined by

$$t_3 - t_2 = \sqrt{[p^3/\mu(1-e^2)^3]}[E_3 - E_2 - e(\sin E_3 - \sin E_2)] \tag{47}$$

where

$$\tan E_2/2 = \sqrt{(1-e)/(1+e)}\tan f_2/2$$

$$\tan E_3/2 = \sqrt{(1-e)(1+e)}\tan f_3/2$$

In total, we have Eqs. (38), (40), (41), (43), and (45–47), which are sufficient to determine the unknowns α_j, φ_i, f_i, θ_i, ψ_i, a_j, C_{2j}, P_j, R_j, B, D, τ_j, p, e, ω, α_j, and t_2, where $i = 1, \ldots, 4$ and $j = 1, 2$, and, without loss of generality, we choose $t_1 = 0$, $\tau_1 = 0$, and $\theta_1 = 0$.

We can now begin the process of solving for the 32 unknowns. At each switching point (see Fig. 2), the unknowns are 1) switching point 1: φ_1, ψ_1, a_1, C_{21}, P_1, f_1, R_1; 2) switching point 2: α_1,φ_2, ψ_2, θ_2, f_2, ω, p, e, t_2, B, D; 3) switching point 3: φ_3, ψ_3, θ_3, f_3, a_2, C_{22}; and 4) switching point 4: α_2, φ_4, ψ_4, θ_4, τ_2, P_2, f_4, R_2.

First, by appropriate choice of the reference line $f = 0$ (see Ref. 1), from the first equation for primer vector at the first switching point, we find that $f_1 = 0$ and $\varphi_1 = 0$. Then, using Eq. (38), and taking into account Eq. (39), we determine that

$$\alpha_1 = \arctan(-F_2/F_1) - (km_0/\beta), \qquad f_1 = \arctan(-\sin2\alpha_1/2)$$

$$\varphi_1 = \alpha_1 + \pi/2, \qquad \dot{\varphi}_1 = k, \qquad r_0 = -C_{21}a_1 \tag{48}$$

and, from Eq. (43) and (44), we obtain

$$\alpha_2 = \arctan(-F_4/F_3) - k_1(m_0-\beta t_2)/\beta$$

$$f_4 = \arctan[-\sin2(k_1\tau_2+\alpha_2)/2], \qquad \varphi_4 = k_1\tau_2 + \alpha_2 + \pi/2$$

$$\dot{\varphi}_4 = k_1, \qquad r_1 = -C_{22}a_2 \tag{49}$$

From Eqs. (40) and (48), it follows that

$$e\cos f_2 = -\frac{p}{r_0}\frac{\cos\varphi_2}{\sin\gamma} - 1 \tag{50}$$

$$\frac{\sqrt{\mu}}{\sqrt{p}}e\sin f_2 = -kr_0\frac{\cos^2\gamma - \sin^2\varphi_2}{\cos\gamma\cos\varphi_2} \tag{51}$$

where $\gamma = kt_2 + \alpha_1$. Substituting Eq. (50) into (40) yields

$$p = r_0\tan^2\gamma\tan^2\varphi_2 \tag{52}$$

from which it follows that

$$e\cos f_2 = -\tan^2\gamma\tan^2\varphi_2\frac{\cos\varphi_2}{\cos\gamma} - 1 \tag{53}$$

$$e\sin f_2 = \tan\gamma\tan\varphi_2\frac{\cos^2\gamma - \sin^2\varphi_2}{\cos\gamma\cos\varphi_2} \tag{54}$$

The expressions in Eqs. (52–54) for p, e, and f_2 are given in terms of γ and φ_2. Equations (41–43) have the same form as Eqs. (38–40); therefore, we can obtain expressions in the form

$$p = r_1\tan^2\alpha_2\tan^2\varphi_3 \tag{55}$$

$$e\cos f_3 = -\tan^2\alpha_2\tan^2\varphi_3\frac{\cos\varphi_3}{\cos\alpha_2} - 1 \tag{56}$$

$$e\sin f_3 = \tan\alpha_2\tan\varphi_3\frac{\cos^2\alpha_2 - \sin^2\varphi_3}{\cos\alpha_2\cos\varphi_3} \tag{57}$$

These three equations allow us to determine p, e, and f_3 as a function of φ_3. Solving for the integration constant B from Eqs. (40) and (41) and equating the resulting expressions leads to

$$\frac{a_1\sin\gamma\sin\varphi_2}{e\sin f_2} = \frac{a_2\sin\alpha_2\sin\varphi_3}{e\sin f_3} \tag{58}$$

Then, substituting

$$a_2 = a_1(\sin\gamma/\sin\alpha_2)$$

into Eq. (58) yields the relationship between the angles

$$\sin f_3/\sin f_2 = \sin\varphi_3/\sin\varphi_2 \tag{59}$$

Equating the expressions (52) and (55) for p, we obtain

$$\tan^2\gamma\tan^2\varphi_2 = \rho\tan^2\alpha_2\tan^2\varphi_3 \tag{60}$$

where $\rho = r_1/r_0$. Substituting $\sin f_3$ obtained from Eq. (59) into Eq. (57) yields

$$e\sin f_2 = \tan\alpha_2\tan\varphi_3\frac{\sin\varphi_2}{\sin\varphi_3}\frac{\cos^2\alpha_2 - \sin^2\varphi_3}{\cos\alpha_2\cos\varphi_3} \tag{61}$$

Equating the expressions in Eqs. (54) and (61) for $e\sin f_2$ and also using the expressions in Eqs. (53), (54), (56), and (57), we obtain

$$\tan\alpha_2\tan\varphi_3\frac{\sin\varphi_2}{\sin\varphi_3}\frac{\cos^2\alpha_2 - \sin^2\varphi_3}{\cos\alpha_2\cos\varphi_3} = \tan\gamma\tan\varphi_2\frac{\cos^2\gamma - \sin^2\varphi_2}{\cos\gamma\cos\varphi_2} \tag{62}$$

$$\left[\tan^2\gamma\tan^2\varphi_2\frac{\cos\varphi_2}{\cos\gamma} + 1\right]^2 + \left[\tan\gamma\tan\varphi_2\frac{\cos^2\gamma - \sin^2\varphi_2}{\cos\gamma\cos\varphi_2}\right]^2 = \left[\tan^2\alpha_2\tan^2\varphi_3\frac{\cos\varphi_3}{\cos\alpha_2} + 1\right]^2 + \left[\tan\alpha_2\tan\varphi_3\frac{\cos^2\alpha_2 - \sin^2\varphi_3}{\cos\alpha_2\cos\varphi_3}\right]^2 \tag{63}$$

From Eqs. (60) and (62), we have

$$\frac{\cos^2\gamma - \sin^2\varphi_2}{\cos\gamma\cos\varphi_2} = d_2\sin\varphi_2 \tag{64}$$

$$\cos^2\gamma = \frac{\tan^2\varphi_2}{\tan^2\varphi_2 + d_1} \tag{65}$$

where d_1 and d_2 are defined as

$$d_1 := \rho \tan^2 \alpha_2 \tan^2 \varphi_3, \qquad d_2 := \frac{1}{\sqrt{\rho}\sin\varphi_3}\frac{\cos^2\alpha_2 - \sin^2\varphi_3}{\cos\alpha_2\cos\varphi_3} \tag{66}$$

Then, from Eqs. (64) and (65), it follows that

$$d_2^2 = \cos^2\gamma\left[(1-d_1)^2\big/d_1\right] \tag{67}$$

From the last expression we obtain

$$\frac{\cos^2\alpha_2 - \sin^2\varphi_3}{\cos\alpha_2\cos\varphi_3} = d_2^2\rho\sin^2\varphi_3 \tag{68}$$

Substitution of Eqs. (64–68) into Eqs. (63) gives the following equation in terms of t_2 and d_1:

$$\left[\frac{d_1}{\cos\gamma}\sqrt{\frac{1}{\tan^2\gamma + d_1}} + 1\right]^2 - \left[\frac{d_1}{\sqrt{\rho}\cos\alpha_2}\sqrt{\frac{1}{\rho\tan^2\alpha_2 + d_1}} + 1\right]^2 + \cos^2\gamma(1-d_1)^2\left(\sin^2\varphi_2 - \sin^2\varphi_3\right) = 0 \tag{69}$$

where

$$\sin^2\varphi_2 = \frac{d_1}{\tan^2\gamma + d_1}, \qquad \sin^2\varphi_3 = \frac{d_1}{\rho\tan^2\alpha_2 + d_1}$$

$$\gamma = kt_2 + \alpha_1 \tag{70}$$

The unknown d_1 can be found by the use of the expressions for the switching function and angles φ_1, φ_2, and φ_4. Indeed, using the expressions for φ_1 and φ_4, we can write

$$\tan\alpha_1\tan\varphi_1 = -1, \qquad \tan(k_1\tau_2 + \alpha_2)\tan\varphi_4 = -1 \tag{71}$$

From the first equations of Eqs. (40) and Eqs. (43), one can obtain

$$F_2 = -F_1(\sin\alpha_3/\cos\alpha_3), \qquad F_4 = -F_3(\sin\alpha_4/\cos\alpha_4) \tag{72}$$

Substitution of Eqs. (71), Eqs. (72), and the first equation of Eqs. (66) into the corresponding formulas for φ_2 and φ_4 leads to

$$d_1 = \left[1 - \frac{cF_1}{kr_0\cos\alpha_3}\right]^2, \qquad \frac{d_1}{\rho} = \left[1 + \frac{cF_3}{k_1r_1\cos\alpha_4}\right]^2 \tag{73}$$

and excluding d_1 results in

$$\rho = \left[\left(1 - \frac{cF_1}{kr_0\cos\alpha_3}\right)\left(1 + \frac{cF_3}{k_1r_1\cos\alpha_4}\right)\right]^2 \tag{74}$$

which contains the unknowns t_2 and τ_2.

Substituting Eqs. (73) into Eqs. (69), one can obtain an equation that also contains the unknowns t_2 and τ_2. Therefore, Eqs. (69) and (74) serve to find t_2 and τ_2 as functions of given quantities

$$t_2 = t_2(\rho, r_0, c, m_0, \beta), \qquad \tau_2 = \tau_2(\rho, r_0, c, m_0, \beta)$$

Note that these functions cannot be written explicitly because Eqs. (69) and (74) contain integral sine and integral cosine, the arguments of which are linear functions of t_2 and τ_2. Once these quantities become known, then all other unknowns can be found in terms of elementary functions as described hereafter.

The constants α_1 and α_2 are computed by the use of Eqs. (48) and (49). The quantity d_1 is found by the use of one of the equations of Eqs. (73). Substituting Eqs. (60) into Eq. (63), we obtain an expression in terms of φ_3 and α_2:

$$\left[\frac{2d_1}{\sqrt{\rho(b_1+d_1)}} + \frac{d_1 - b_1}{\rho} + d_1\right]b_2^2 - 2\sqrt{d_1}b_2 - 2d_1 = 0 \tag{75}$$

where

$$b_1 = (\rho - d_1)\sin^2\alpha_2, \qquad b_2 = \sqrt{\frac{b_1(r - d_1)}{d_1\rho^2(1-d_1)} + 1}$$

Between $\tan\varphi_3$, d_2, and d_1, we have the following relationships:

$$\tan^2\varphi_3 = \frac{d_1}{\rho\tan^2\alpha_2} \tag{76}$$

$$d_2^2 = \frac{(\rho - d_1)^2}{d_1\rho^2}\sin^2\alpha_2 \tag{77}$$

Equation (75) yields complex roots if $d_1 < 1$, which, as follows from Eqs. (52) and (55), corresponds to the inequalities $d_1 < 1$ and $p < r_0$ or $p > r_1$. Therefore, the value of d_1 from Eq. (75) should satisfy the condition

$$1 < d_1 < \rho \tag{78}$$

To exclude complex valued solutions, it follows from Eqs. (68) and (75) that we must satisfy the following relationships:

$$d_1 \neq \rho \tag{79}$$

$$d_1 - d_{11} > 0, \qquad d_1 - d_{12} > 0$$

or

$$d_1 - d_{11} < 0, \qquad d_1 - d_{12} < 0 \tag{80}$$

where

$$d_{11} = \frac{1 - (1/\rho)\sin^2\alpha_2 - [(1-\rho)/\rho]\sin\alpha_2}{1 - (1/\rho^2)\sin^2\alpha_2}$$

$$d_{12} = \frac{1 - (1/\rho)\sin^2\alpha_2 + [(1-\rho)/\rho]\sin\alpha_2}{1 - (1/\rho^2)\sin^2\alpha_2}$$

Thus, the transfer between concentric circular orbits in the central Newtonian field using two MT arcs is possible if the boundary orbits satisfies the inequalities in Eqs. (78–80).

If d_1 is known, then the other solutions to the problem can be obtained as follows. The thrust angles at the second and third junctions are found from Eqs. (66) as

$$\tan\varphi_2 = \sqrt{d_1/\tan^2\gamma}, \qquad \tan\varphi_3 = \sqrt{d_1/\rho\tan^2\alpha_2}$$

Then, from Eqs. (52–57), we obtain the eccentricity e, semilatus rectum p, and angles f_2 and f_3:

$$p = r_0\tan^2\gamma\tan^2\varphi_2$$

$$e^2 = \left[\frac{d_1}{\cos\gamma}\sqrt{\frac{1}{\tan^2\gamma + d_1}} + 1\right]^2 + \cos^2\gamma(1-d_1)^2\sin^2\varphi_2$$

$$\tan f_2 = -\left[\sqrt{d_1}d_2\sin\varphi_2\bigg/\left(d_1\frac{\cos\varphi_2}{\cos\gamma} + 1\right)\right]$$

$$\tan f_3 = -\left[\sqrt{d_1}d_2\sin\varphi_3\bigg/\left(\frac{d_1}{\rho}\frac{\cos\varphi_3}{\cos\alpha_2} + 1\right)\right]$$

where d_2 is computed using Eqs. (66). The true anomalies and ψ angles are found by the use of the expressions

$$\psi_1 = (\pi/2) - \varphi_1 = -\alpha_1, \qquad \psi_2 = \psi_1$$

$$\psi_3 = \theta_3 - \varphi_3 + (\pi/2), \qquad \psi_4 = \psi_3$$

$$\theta_2 = \psi_2 + \varphi_2 - (\pi/2), \qquad \theta_3 = \theta_2 + f_3 - f_2$$

$$\theta_4 = \psi_4 + \varphi_4 - \pi/2$$

The transversality condition given in Eq. (45) can be used to find a_2:

$$(c/\beta)a_2\sin(k_1\tau_2 + \alpha_2) = m_0/\beta - (t_2 + \tau_2) \tag{81}$$

The integration constants a_1, C_{21}, and C_{22} obtained using Eqs. (41), (48), and (49) and the continuity condition for the magnitude of the primer vector at the second and third switching points

$$a_1 \sin(kt_2 + \alpha_1) = a_2 \sin\alpha_2$$

are found to be

$$a_1 = \frac{a_2 \sin\alpha_2}{\sin(kt_2 + \alpha_1)}, \qquad C_{21} = -\frac{r_0}{a_1}, \qquad C_{22} = -\frac{r_1}{a_2} \tag{82}$$

The value of λ_0 can be found as

$$\lambda_0 = a_1 \sin\alpha_1 \tag{83}$$

Consequently, Eqs. (40), (41), (43), and (46) yield the relationships

$$2\tan f_4 = \sin 2\varphi_4, \qquad R_1 = -a_1 \sin\alpha_1$$

$$R_2 = -a_2\cos^2\varphi_4 + 2a_2, \qquad B = \frac{a_1 \sin\gamma \sin\varphi_2}{e \sin f_2}$$

$$D = -B\left(d_1\frac{\cos\varphi_2}{\sin\gamma}\right)^2 - a_1\sin\gamma\cos\varphi_2\left(d_1\frac{\cos\varphi_2}{\sin\gamma}\right)^2$$

$$P_1 = a_1\sin\alpha_1, \qquad P_2 = a_2^2(1+\sin^2\varphi_4\cos^2\varphi_4) \tag{84}$$

The multiplier λ_7 has the following values according to the condition $\chi = 0$ at all switching points:

$$\lambda_7 = \frac{ca_1\sin\alpha_1}{m_0} \quad \text{at} \quad t = t_1 = 0$$

$$\lambda_7 = \frac{ca_1\sin(kt_2+\alpha_1)}{m_0 - \beta t_2} \quad \text{at} \quad t = t_2$$

$$\lambda_{71} = \frac{ca_2\sin\alpha_2}{m_0 - \beta t_2} \quad \text{at} \quad \tau = \tau_1 = 0$$

$$\lambda_{71} = \frac{ca_2\sin(k_1\tau_2+\alpha_2)}{m_0 - \beta(t_2+\tau_2)} \quad \text{at} \quad \tau = \tau_2 \tag{85}$$

Using Eq. (47) and the expressions valid at the second and third switching points, namely,

$$\tan(E_2/2) = \sqrt{(1-e)/(1+e)}\tan(f_2/2)$$

$$\tan(E_3/2) = \sqrt{(1-e)/(1+e)}\tan(f_3/2)$$

we obtain an expression to compute the time interval of the motion along the Keplerian transfer orbit. The time of the NT arc is

$$T_{\rm NT} = \sqrt{[p^3/\mu(1-e)^3]}[(E_3 - E_2) - e(\sin E_3 - \sin E_2)] \tag{86}$$

The gravitational errors at the second and third switching points due to the linear field approximation are determined by formulas

$$|\epsilon_2| = \left|1/r^3(f_2) - 1/r_0^3\right|, \qquad |\epsilon_3| = \left|1/r_1^3 - 1/r^3(f_3)\right| \tag{87}$$

where

$$r(f_2) = p/(1+e\cos f_2), \qquad r(f_3) = p/(1+e\cos f_3)$$

To evaluate the relative error $\mathcal{O}(\Delta r/r_0)$ in the performance functional and the relative error $\mathcal{O}(\Delta r/r_0^4)$ in the dynamic model, we use the following formulas at the second and third switching points:

$$\mathcal{O}[\Delta r(f_2)/r_0)] = [r(f_2) - r_0]/r_0$$

$$\mathcal{O}[\Delta r(f_3)/r_1] = [r_1 - r(f_3)]/r_1$$

$$\mathcal{O}\left[\Delta r(f_2)/r_0^4\right] = [r(f_2) - r_0]/r_0^4$$

$$\mathcal{O}\left[\Delta r(f_3)/r_1^4\right] = \left[r_1 - r(f_3)/r_1^4\right]$$

In the next section, we solve an orbit transfer problem between concentric circular orbits and show the results of the numerical computations using the analytic solutions contained in Eqs. (48–87).

Numerical Example

We now consider the transfer between concentric circular orbits in a central Newtonian field (around the Earth with $\mu = 398{,}600.1445$ km^3/s^2). Suppose that we desire to transfer a spacecraft with a propulsion system with constant exhaust velocity $c = 3$ km/s from an initial orbit with radius of $r_0 = 6600$ km to a final orbit with radius $r_1 = 26{,}400$ km. This selection of r_0 and r_1 is chosen only to demonstrate the solution process, and other radii can be considered. Thus, for the given r_0 and r_1, we have $\rho = r_1/r_0 = 4$, $k = 0.0011775$ rad/s, and $k_1 = 0.00014718$ rad/s. We assume that $m_0 = 10{,}000$ kg and $\beta = m_0/100$ kg/s. The resulting orbital transfer trajectory, including the two MT arcs, is illustrated in Fig. 3.

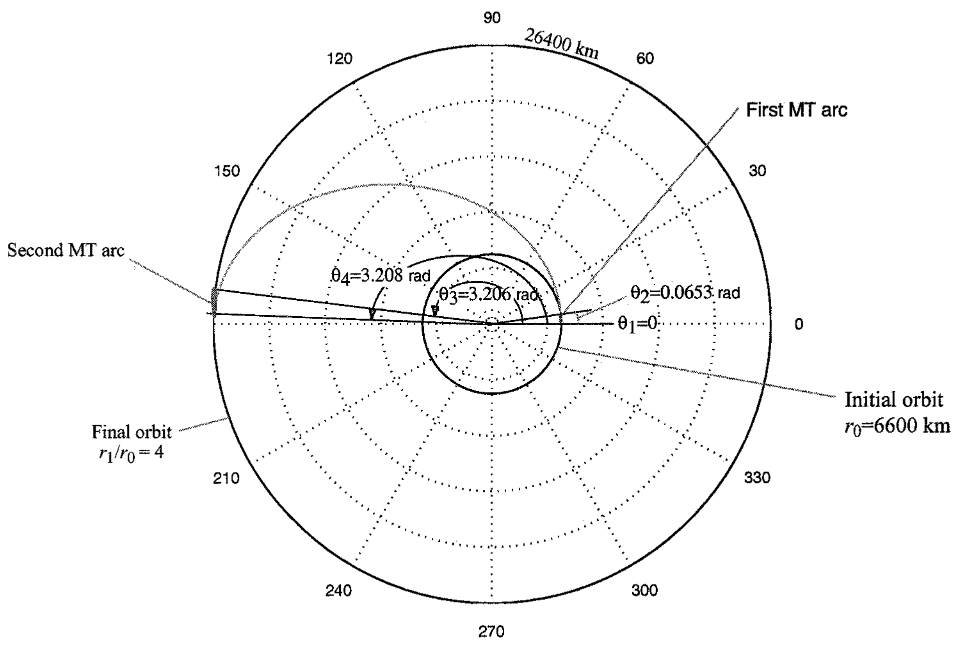

Fig. 3 Transfer trajectory including two MT arcs.

When the various analytic solutions are utilized, the 32 unknowns (listed according to switching point) are computed to be as follows.

Switching point 1:

$$\varphi_1 = -0.03256, \quad \psi_1 = 1.60336, \quad a_1 = 1042.6035$$

$$C_{21} = -6.3303, \quad f_1 = -0.03253$$

$$P_1 = -1042.6023, \quad R_1 = 1042.6023$$

Switching point 2:

$$\alpha_1 = -1.60336, \quad \varphi_2 = 0.03279, \quad \psi_2 = 1.60336$$

$$\theta_2 = 0.06535, \quad f_2 = 0.00085, \quad \omega = 0.0645$$

$$p = 10{,}561.7006, \quad e = 0.59993, \quad t_2 = 49.67158$$

$$B = -66{,}988.3772, \quad D = -169{,}809.6909$$

Switching point 3:

$$\varphi_3 = -0.00096, \quad \psi_3 = 4.77782, \quad \theta_3 = 3.20607$$

$$f_3 = 3.14157, \quad a_2 = -1042.2544, \quad C_{22} = 1.0003$$

Switching point 4:

$$\alpha_2 = -1.57231, \quad \varphi_4 = 0.0013, \quad \psi_4 = 4.77782$$

$$\theta_4 = 3.20832, \quad \tau_2 = 19.06081, \quad f_4 = 0.7854$$

$$P_2 = -1042.2553, \quad R_2 = -1042.2562$$

Computing d_1 yields $d_1 = 1.600257673$, which satisfies the inequality in Eq. (78); therefore, we know that a transfer using MT arcs exists. The time of the transfer between MT arcs is computed via Eq. (86) as $T_{\text{NT}} = 10546.26$ s. The dimensions of the thin spherical layers during the two MT arcs are $\Delta r_2 = |r(f_2) - r_0| = 1.3307$ km and $\Delta r_3 = |r_1 - r(f_3)| = 0.01814$ km, respectively. Errors in the minimizing functional and dynamic model, associated with the approximation, are as follows:

$$\mathcal{O}[\Delta r(f_2)/r_0] = [r(f_2) - r_0]/r_0 = 2.02e - 04$$

$$\mathcal{O}[\Delta r(f_3)]/r_0 = [r_1 - r(f_3)]/r_1 = 6.87e - 07$$

$$\mathcal{O}\big[\Delta r(f_2)/r_0^4\big] = \big[r(f_2) - r_0\big]/r_0^4 = 7.03e - 16$$

$$\mathcal{O}\big[\Delta r(f_3)/r_0^4\big] = \big[r_1 - r(f_3)\big]/r_1^4 = 3.73e - 20$$

These are small enough to justify the linearity assumption of the central gravity field during the MT arcs. Utilizing the magnitudes of the radius vector, one can evaluate the relative error associated with neglecting higher-order terms in the expansion of quantity μ/r_0^3 [see Eq. (3)] at the second and third switching points:

$$|\epsilon_2| = 2.10e{-}15 \text{ 1/km}^3 \text{(at the second switching point)}$$

$$|\epsilon_3| = 1.120e{-}19 \text{ 1/km}^3 \text{(at the third switching point)}$$

The magnitude of the velocity increment caused by an expenditure of propellant is $\Delta V = c \ln(m_0/m_f)$. In the case of nonvanishing external forces, if the exhaust velocity is constant, then ΔV can be treated as a convenient measure of fuel expenditure. It is known as the characteristic velocity.[1] Under the orbit conditions in our example, the impulsive Hohmann transfer maneuver yields

$$\Delta V_{\text{IMP}}/v_0 = [1 - (1/\rho)]\sqrt{2\rho/(1+\rho)} + \sqrt{1/\rho} - 1 = 0.44868$$

For comparison, using the analytic solutions we compute the ratio of characteristic velocity to the initial velocity to be

$$\Delta V_{\text{MT}}/v_0 = 0.448797$$

When implementing impusive solutions on realistic rockets with finite mass flow rate, it is necessary to determine practical engine-on times. There are many ways this can be done, and it is not the purpose of this paper to address the various techniques available. A separate study would be required to compare the analytic solutions described herein with the various real-world implementations of the impulsive solutions. However, what we can do here is to determine the errors incurred by utilizing the analytic solution (based on a linear field approximation) in a central gravity field. This is accomplished by employing numerical integration methods.

In the numerical experiments shown in Figs. 4–11, we integrate the spacecraft equations of motion utilizing the thrust direction time history obtained analytically. The numerical integration was implemented on a Cray SV1 machine using International Mathematical and Statistical Library subroutines employing a fifth-order fixed step Runge–Kutta algorithm. The maximum absolute error for all computations was 10^{-9}.

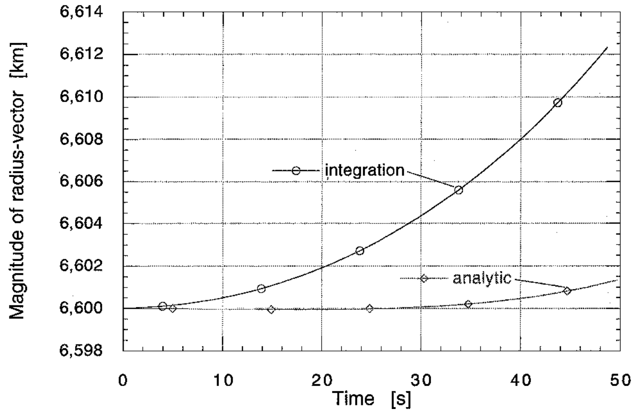

Fig. 4 First MT arc: magnitude of radius vector vs time.

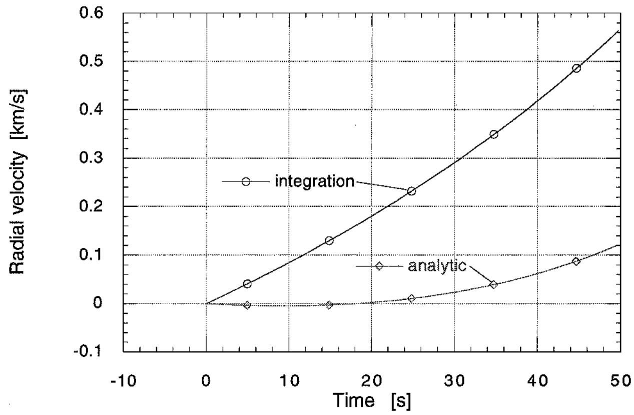

Fig. 5 First MT arc: magnitude of radial velocity vs time.

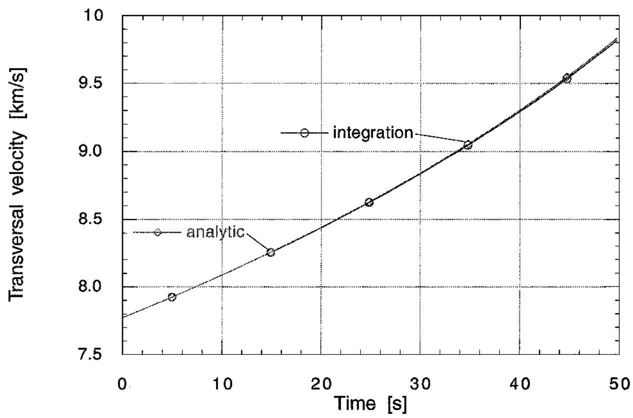

Fig. 6 First MT arc: magnitude of transversal velocity vs time.

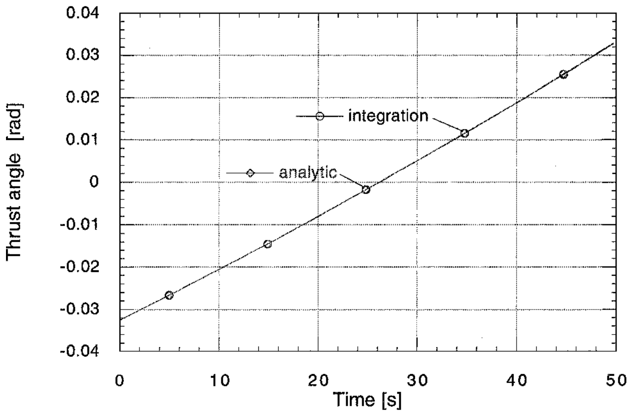

Fig. 7 First MT arc: thrust angle as function of time.

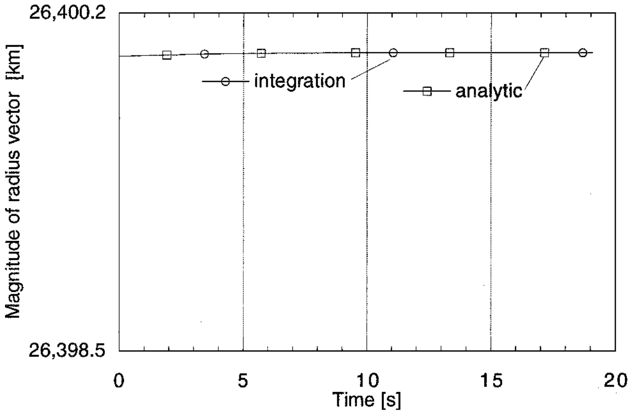

Fig. 8 Second MT arc: magnitude of radius vector vs time.

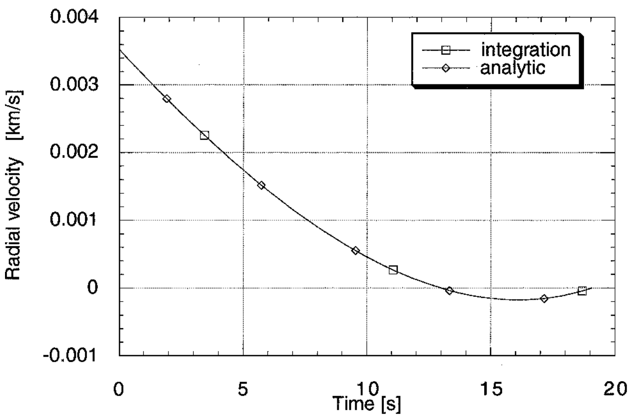

Fig. 9 Second MT arc: magnitude of radial velocity vs time.

Figures 4–7 describe the time histories of the position magnitude, velocity magnitude, and thrust angle (with respect to the local horizontal) during the first MT arc. The plots compare the analytic results with the numerical integration. As shown in Fig. 4, the error in position magnitude at the end of the first MT arc is approximately 11.0 km. From Figs. 5 and 6, it can be seen that the total velocity error at the end of the first MT arc is about 4%. Figure 7 shows the differences in the thrust direction. Note from Figs. 7 and 11 that the thrust angle is less than 1.87 deg, which is consistent with the linearity assumption [see Eq. (9)]. Note that the thrust angle φ, depicted in Fig. 7, is with respect to the local horizontal and not with respect to the inertial reference axis where $\theta = 0$ (see Fig. 1). As we proved, the angle ψ with respect to the inertial reference axis ($\theta = 0$) is constant. For the second MT arc, the results depicted in Figs. 8–11 show similar trends to the first MT arc.

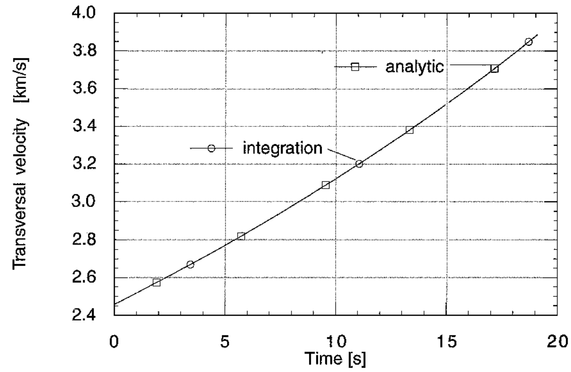

Fig. 10 Second MT arc: magnitude of transversal velocity vs time.

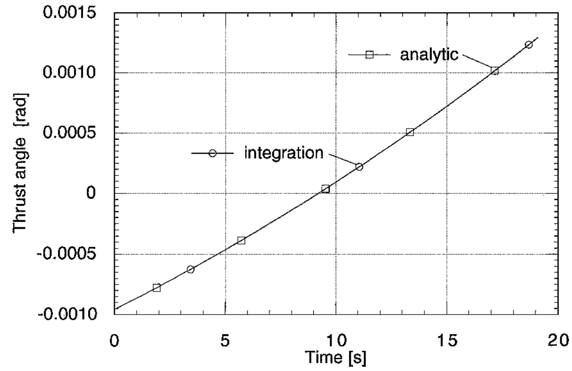

Fig. 11 Second MT arc: thrust angle as function of time.

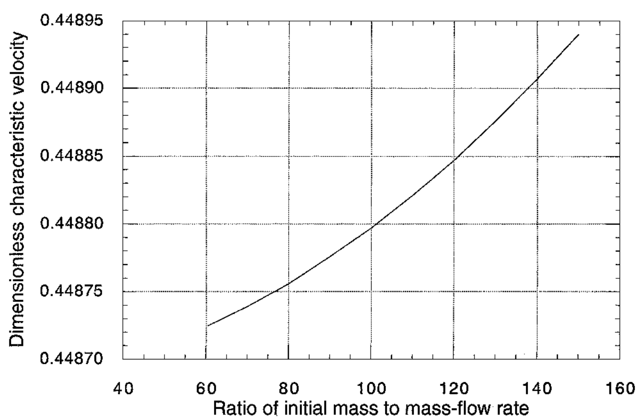

Fig. 12 Relationship between ΔV_{MT} and ratio m_0/β.

Analyses of the analytic solutions obtained for different initial conditions indicate that the characteristic velocity is sensitive to the values of $\rho = r_1/r_0$ and is not highly sensitive to the values of m_0/β for given terminal orbits and exhaust velocity. The characteristic velocity vs the ratio of the terminal radii that satisfy the inequality $2 \leq \rho \leq 10$ and the mass flow rate that satisfies the inequality $m_0/150 \leq \beta \leq m_0/60$ (which corresponds to the inequality $2g \leq T/m_0 \leq 5g$) are given in Figs. 12 and 13, which show the sensitivity of the ΔV to the ratios $\rho = r_1/r_0$ and m_0/β.

It was found that the dimensionless $\Delta V_{MT}/v_0$ differs from the $\Delta V_{IMP}/v_0$ of the impulsive Hohmann transfer by at most 0.0005 for the given ρ from $2 \leq \rho \leq 10$ (see Fig. 14). Note that the impulsive Hohmann transfer implies that $\beta = \infty$ (Ref. 1). From these results and analyses, one can conclude that fuel expenditure for the orbital transfer being considered in the case of using the MT arcs $\beta = \text{const}$ is comparable to the impulsive transfer $\beta = \infty$ depending on initial conditions and the characteristics of the engine system. Note that using MT arcs allows one to analyze time histories of all parameters for different cases of terminal conditions (while the engine system is turned on) without relying on numerical integration schemes.

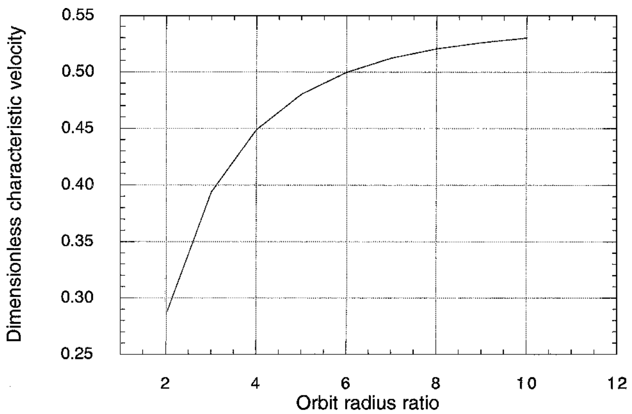

Fig. 13 Relationship between ΔV_{MT} and orbit radius ratio.

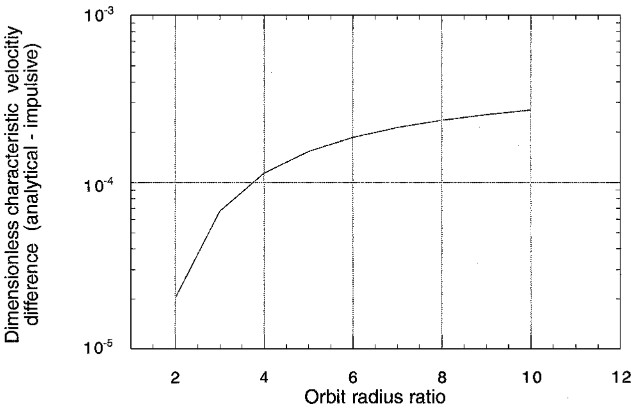

Fig. 14 Difference $\Delta V_{MT} - \Delta V_{IMP}$ vs orbit radius ratio.

One final observation in using the analytic solutions with various initial conditions is that the eccentricity of the transfer orbit increases as the radii are increased. It can be shown that this dependency is similar to the same dependency obtained from the Hohmann transfer case.[4]

Conclusions

In this paper, the Mayer variational problem of computing optimal trajectories of a rocket moving with MT (constant exhaust velocity and limited mass flow rate) in a central Newtonian field has been considered. It was shown that this class of problems admits an approximation of the central Newtonian field by a linear central field when the thrust vector is nearly aligned with the local horizon and the rocket remains in a thin spherical layer during the maneuver. Analytic closed-form solutions for the MT arcs in the linear central field have been obtained. It is proved that when the flight time is not specified, the thrust direction is inertially fixed. The free time minimum fuel problem of orbit transfer between coplanar concentric circular orbits was used to illustrate the solution process comprising two MT and three NT arcs (including the initial and final orbits). In particular, the analysis shows that the numerical results obtained using MT arcs are comparable with the characteristics of impulsive maneuvers. When viewed as a method for targeting orbit maneuvers, assuming practical propulsion system characteristics, that is, without assuming impulsive maneuvers, the analytic solution yields the rocket engine-on/-off times and the thrust magnitude and direction. Therefore, the solutions obtained for thrust direction may be useful as a zeroth-order approximation in the development of a rocket engine steering law applicable in a real gravity field.

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